

# Inhomogeneous Heisenberg Spin Chain as a Non-Holonomically Deformed NLS System

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## Abstract

The Heisenberg Spin Chain system, in the continuum limit, can be represented by the non-linear Schrödinger equation through the Hashimoto map. Inhomogeneity induced through localizing the nearest neighbor interaction strength can also be mapped similarly to an integro -differential generalization of the non-linear Schrödinger system [J. Phys. C: Solid State Phys. **15**, L1305 (1982)] which is integrable. We show that the latter system is a particular non-holonomic deformation of the usual non-linear Schrödinger equation, aided by generalized parameterizations. General non-holonomic deformations correspond only to temporal inhomogeneity, with the additional spatial inhomogeneity corresponding to particular spectral orders, identical for both continuum Heisenberg spin chain and non-linear schrödinger system. Being semi-Classical in nature, these continuum limits of spin system display amplitude-phase entanglement.

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# 1 Introduction

The Heisenberg spin chain (HSC) [1] is one of the most celebrated concepts in both condensed matter theory, being the simplest description of spin-dependency of magnetization, which can be extended to incorporate temperature effects on the latter. From an analytic view-point, being the simplest interacting spin structure, constituted by nearest neighbor interactions,

$$H = -J \sum_{i=1}^N \mathcal{S}_i \cdot \mathcal{S}_{i+1}, \quad (1)$$

its solvability garners practical insights.  $\mathcal{S}_i$  represents the spin vector in sight  $i$  and thereby the above Hamiltonian represents nearest neighbor spin-spin interaction of strength  $J$ . The sign of the constant  $J$  determines ferromagnetic ( $J > 0$ ) or anti-ferromagnetic ( $J < 0$ ) nature of the systems. Although being spatially unidimensional, the vector nature of the individual variables in the above Hamiltonian originates from the three-dimensional internal  $SU(2)$  sub-space of spin-orientations. This prompts the designation of XXX model to the above system. Breaking of isotropy along the three spin-axis in the  $SU(2)$  subspace leads to XXZ and XYZ models respectively.

This XXX HSC can be mapped to a non-linear Schrödinger (NLS) system. For that purpose, one considers the corresponding Heisenberg equation of motion (EOM),

$$\mathcal{S}_{i,u} = J \mathcal{S}_i \times \sum_j \mathcal{S}_j \quad (\hbar = 1), \quad (2)$$

where  $u$  stands for the temporal coordinate and  $s$  will represent the spatial one in the following. In the *continuum limit*, with lattice separation  $a \rightarrow 0$ , it yields the semi-classical EOM [3],

$$\mathbf{t}_u = \mathbf{t} \times \mathbf{t}_{ss}, \quad (3)$$

with the normalization  $Ja^2 = 1$  and  $\mathbf{t}$  representing the *Classical* analogue of  $\mathcal{S}$ . The abbreviations  $u$  and  $s$  can further be considered to define a curve in a manifold with  $\mathbf{t}$  representing the tangent vector to that curve in a Frenet-Serret basis. This enables the curvature ( $\kappa$ ) and torsion ( $\tau$ ), defined in that basis, identified with respective energy and momentum densities  $\kappa^2$  and  $\kappa^2\tau$  [4]. Finally, through the implication of the Hasimoto transformation,

$$q(s, u) = \kappa(s, u) \exp \left\{ i \int_{-\infty}^s \tau(s', u) ds' \right\}, \quad (4)$$

which was initially obtained for generic vortex filaments [5], the XXX HSC can be mapped, in the continuum, semi-classical limit, to the standard NLS system satisfied by the continuous variable  $q$  [4] as:

$$q_u - iq_{ss} + 2i\eta|q|^2q = 0. \quad (5)$$

The NLS system is very well understood, and generalizes into many extended solvable [2] and quasi-solvable [7] models, representing real physical systems. Usually it represents mean-field dynamics of many-particle systems, and thereby the above correspondence with the HSC potentially serves as a connection between two different classes of semi-classical physical systems. Further, the characteristic semi-classical solitonic excitations of the NLS system directly corresponds to those observed in the XXX HSC. The case of inhomogeneous HSC, with site ( $i$ ) dependent  $J \rightarrow J_i$  in the discrete case still preserves isotropy in the  $SU(2)$  space, thereby yielding a *inhomogeneous* XXX model. Such a model was shown to be represented by a integro-differential extension of the NLS system [8], which is integrable [9] with a geometric interpretation [10]. However, such a system cannot be accessed by deforming the standard NLS system, owing to the above-mentioned constraints.

In the present work, we show that a nonholonomic (NH) extension of the NLS system can correspond to local (both in space and time) generalization of the ferromagnetic coefficient  $J \rightarrow J(x, t)$  in the continuum limit, with the spatial dependence arising out of NH contribution from a very specific spectral reason, represented by zeroth and first power of the spectral parameter  $\lambda$ . In that case, the usual additional NH constraint structure manifests purely in terms of the undeformed NLS variables. Otherwise, the modified NLS system only shows temporal inhomogeneity that corresponds only to time-dependence of the ferromagnetic coefficient, with NH constraints independent of the NLS parameters. In any case, the deformed NLS system exhibit amplitude-phase interdependence, absent in pure NLS system, as a strong signature of the quantum nature of the original deformation in HSC.

In the following, we will discuss the NH deformation of the XXX model at the continuum level in Section 2, displaying splitting of the spectral hierarchy of the deformation imposed by dynamics of the system. It will be followed by NH deformations of the usual NLS system in the subsequent Section 3, explicitly showing when the latter can represent temporal as well as spatial inhomogeneity. The precise spectral range will be highlighted for which the exact Hasimoto transform of the inhomogeneous XXX HSC [8] is obtained, coinciding with the spectral splitting in case of continuum HSC. Also, the amplitude-phase dependence will be obtained, signifying the semi-classical origin across the Hasimoto transformation. In the next section, we will summarize the results with emphasis on possible introduction of *anisotropy* to the XXX model in the  $SU(2)$  subspace (thereby achieving XXZ or XYZ models) and related issues. Then we will conclude.

## 2 NH Deformation of The HSC

The semi-Classical EOM for the HSC in Eq. 3 can more conveniently be expressed in the matrix form as:

$$S_u = \frac{1}{2i} [S, S_{ss}], \quad (6)$$

where  $S$  is the trace-less matrix representation,

$$S = \begin{pmatrix} t_3 & t_1 - it_2 \\ t_1 + it_2 & -t_3 \end{pmatrix} \equiv t_i \sigma_i, \quad (7)$$

of the semi-classical limit of the spin vector in a *new*  $SU(2)$  subspace,  $\mathbf{t}$ , having components  $t_i$ ,  $i = 1, 2, 3$ . Here,  $\sigma_i$  are the three Pauli matrices. It is to be noted that *after* the semi-classical limit has been implemented to arrive at Eq. 3, the vector  $\mathbf{t}$  exists in the three-dimensional coordinate space, as the tangent to a curve parameterized by  $(s, u)$ . Eq. 7 introduces a new  $SU(2)$  basis to represent  $\mathbf{t}$  in a more convenient way. Thus, it is totally classical and no quantum nature is being implemented by it. The Lax pair corresponding to the solvable system of Eq. 6 is given as [?],

$$U = i\lambda S, \quad V = 2i\lambda^2 S - \lambda S_s S, \quad (8)$$

which engerates space ( $\Phi_u = U\Phi$ ) and time ( $\Phi_s = V\Phi$ ) evolutions respectively. The corresponding curvature vanishes,

$$F_{us} := U_u - V_s + [U, V] = 0, \quad (9)$$

ensuring integrability of the system.

## 2.1 NH Deformation

The simplest non-holonomic deformation of the system can be achieved through the most general extension of the temporal Lax component as,

$$V \rightarrow V_d = V + \delta V, \quad \delta V = \frac{i}{2} \sum_n \lambda^n \boldsymbol{\alpha}^{(n)} \cdot \boldsymbol{\sigma}, \quad n \in \mathbb{I}. \quad (10)$$

To prevail integrability, we impose vanishing of the corresponding modified curvature:  $F_{us} + \delta F_{us} = 0$ . As usual,

$$F_{us} = i\lambda \left( S_u - \frac{1}{2i} [S, S_{ss}] \right), \quad (11)$$

which is  $\mathcal{O}(\lambda^1)$ . The contribution from the deformation is,

$$\delta F_{us} = -\frac{i}{2} \sum_n \lambda^n \boldsymbol{\alpha}_s^{(n)} \cdot \boldsymbol{\sigma} - i \sum_n \lambda^{n+1} \left( \mathbf{t} \times \boldsymbol{\alpha}^{(n)} \right) \cdot \boldsymbol{\sigma}. \quad (12)$$

Thus, on comparing Eq.s 11 and 12, only for the values  $n = 0, 1$ , the EOM gets deformed as,

$$S_u = \frac{1}{2i} [S, S_{ss}] + \frac{1}{2} \Lambda_s^{(1)} - i \left[ S, \Lambda^{(0)} \right], \quad \Lambda^{(n)} := \boldsymbol{\alpha}^{(n)} \cdot \boldsymbol{\sigma}. \quad (13)$$

For all other values of  $n > n_{\min}$ , generic recursive constraints of the form,

$$\Lambda_s^{(n)} - i \left[ S, \Lambda^{(n-1)} \right] = 0; \quad n \neq 0, 1, \quad (14)$$

in addition to,

$$\Lambda_s^{(n_{\min})} = 0. \quad (15)$$

Thus the lowest order deformation is always by an *space-independent* parameter. It will later be seen that this property becomes crucially related to the deformation of the corresponding NLS system, obtained through the Hasimoto transformation. For now, considering the ‘classical’  $SU(2)$  basis, the vector  $\alpha^{(n_{\min})}$  very well can be chosen as a suitable linear combinations of the other two Frenet-Serret basis vectors, as the components of  $\sigma$  forms the basis. Then all the higher components up to  $\alpha^{(0)}$  can be solved recursively by utilizing Eq. 14. To solve for  $\alpha^{(1)}$ , the deformed EOM of Eq. 13 has to be utilized. Given the deformed solution  $S$  is not known a priori, this can only be done by solving Eq. 14 for a pre-determined value of  $\alpha^{(n>1)}$  and then recursively approaching the EOM. Therefore, a completely solvable NH deformation is guaranteed only by a deformation parameter for any particular order  $n > 1$ , since the deformation spectrum ( $n$ ) is ‘bounded’ from below as far as spatial dependence is concerned. Further, there is a bifurcation in the spectral hierarchy about the spectral order of the EOM (*i.e.*,  $n = 1$ ).

### 3 Inhomogeneous HSC: The NLS comparison

An inhomogeneous generalization of the XXX model, with on-site variation of the coupling (ferromagnetic) constant,

$$H' = -J \sum_i \eta_i \mathcal{S}_i \cdot \mathcal{S}_{i+1}, \quad (16)$$

was later found [8] to yield a locally modified, integro-differential, extended NLS model,

$$q_t - i(\eta q)_{xx} + 2iq \int_{-\infty}^x \eta |q|_{x'}^2 dx' = 0. \quad (17)$$

From now on we switch from the  $(s, u)$  manifold with the Frenet-Serret basis to the coordinate space defined by  $(x, t)$  for the NLS dynamics, following exclusive dependence of the system on the tangent  $\mathbf{t}$ . In order to see the connection between the usual and the modified NLS systems, let us consider the Lax pair for NLS system [7],

$$\begin{aligned} A &= -i\lambda\sigma_3 + \rho^* q^* \sigma_+ + \rho q \sigma_- \quad \text{and} \\ B &= i(\lambda^2 + 2\eta|q|^2)\sigma_3 - (\lambda\rho^* q^* + i\rho^* q_x^*)\sigma_+ - (\lambda\rho q - i\rho q_x)\sigma_-, \end{aligned} \quad (18)$$

in the  $sl(2)$  representation, with the constraint  $\eta = |\rho|^2$ . We consider these parameters to be spacio-temporally local for the generalization. The zero-curvature condition,

$$A_t - B_x + [A, B] = 0, \quad (19)$$

leads to the independent relations for each of the  $SU(2)$  generator as,

$$\begin{aligned}
\sigma_+ : \quad & \rho^* (q_t^* + i q_{xx}^* - 2i\eta |q|^2 q^*) + (\rho_t^* q^* + i \rho_x^* q_x^*) = 0, \\
\sigma_- : \quad & \rho (q_t - i q_{xx} + 2i\eta |q|^2 q) + (\rho_t q - i \rho_x q_x) = 0, \\
\sigma_3 : \quad & \eta_x |q|^2 = 0,
\end{aligned} \tag{20}$$

at  $\mathcal{O}(\lambda^0)$  and,

$$\begin{aligned}
\sigma_+ : \quad & \rho_x^* q^* = 0 \quad \text{and} \\
\sigma_- : \quad & \rho_x q = 0,
\end{aligned} \tag{21}$$

at  $\mathcal{O}(\lambda^1)$ . It is clear that the second bracket of the first two of Eq.s 20 arise solely due to locality of the ‘coupling’ parameter  $\eta$ . However, this locality is restricted to a *temporal* one, following the last of Eq.s 20 and Eq.s 21. Following Eq.s 16 and 17, this corresponds to time-dependence of the ferromagnetic parameter  $\eta$ , thereby physically representing a system with change (or even flip) in its magnetic nature over time. This can also be thought as a homogeneous spin-system in a time-dependent temperature field, which essentially changes the magnetization of the system over time.

In order to get the desired non-local dynamics, on imposing the condition,

$$\rho_t q = i \rho (1 - \eta) q_{xx}; \quad \eta = |\rho|^2, \tag{22}$$

one immediately gets the *modified* NLS equation,

$$q_t - i (\eta q)_{xx} + 2i\eta |q|^2 q = 0, \tag{23}$$

which, on identifying the constraint in the third equation of Eq.s 20, can be casted as,

$$q_t - i (\eta q)_{xx} + 2iq \int_{-\infty}^x \eta |q|_x^2 dx' = 0, \tag{24}$$

which is the desired form, simplified from that in Eq. 17 of Ref.s [8, 10]. Such a result can be interpreted in terms of the deformation of Frenet-Serret curve  $\gamma(s)$  [11] across the Hasimoto transformation, with the identification  $\gamma = i|\rho|^2 q_x$ . The complex-conjugation of the above equation leads to the dual sector with variable  $q^*$ . In fact, Eq. 22 can be cast into a more convenient form,

$$\begin{aligned}
& \eta_t |q|^2 = i \eta (1 - \eta) (q^* q_{xx} - q q_{xx}^*) \\
\text{or,} \quad & \left[ \log \left( \frac{\eta}{1 - \eta} \right) \right]_t = i \left( \frac{q_{xx}}{q} - \frac{q_{xx}^*}{q^*} \right) \equiv \eta^{-1} \frac{|q|_t^2}{|q|^2},
\end{aligned} \tag{25}$$

with the last equality coming from the EOM. The above equation expresses  $\eta$  in terms of the NLS amplitude, with the former being *explicitly* real. Interestingly, as  $\eta = \eta(t)$ , the above relation can be true only if all the terms are dependent *only* on time. Thus, on

considering the space-derivative of the middle term above to vanish, following a simple manipulations, one has,

$$\frac{|q|_x^2}{|q|^2} = \frac{(q^* q_x - q q_x^*)_x}{q^* q_x - q q_x^*}. \quad (26)$$

On parameterizing the amplitude as  $q = |q| \exp \{i\theta(x, t)\}$ , the above equation leads to the identification,

$$\theta(x, t) \equiv \frac{i}{2} \mu(t) \int^x \frac{dx'}{|q(x', t)|^2} \int^{x'} dx'' |q(x'', t)|^2, \quad (27)$$

with  $\mu(t)$  is an arbitrary parameter of integration. The above equation can be traced-back to Eq. 22, which has no analogue in the usual NLS system with  $\eta$  being constant. Therefore such an *entanglement* of the amplitude and phase is a signature of the temporal inhomogeneity, and very well can have semi-classical origin rooted at the XXX model.

### 3.1 Deformed XXX Model as a NH Deformation of NLS System

In order to obtain the NLS-like equivalent system for the inhomogeneous XXX model as a NH deformation of the usual NLS system, Let us consider the NH deformation of the latter, achieved through deforming the temporal Lax component as [14],

$$B \rightarrow B + \delta B, \quad \delta B = \frac{i}{2\lambda} [g_3 \sigma_3 + g_+ \sigma_+ + g_- \sigma_-]. \quad (28)$$

We further consider the self-coupling  $\eta$  being a local variable in general. Then, from the zero curvature condition, the coefficients of different  $SU(2)$  generators take the forms:

$$\begin{aligned} \sigma_+ : \quad & \rho^* (q_t^* + i q_{xx}^* - 2i\eta |q|^2 q^*) + \rho_t^* q^* + \frac{1}{2} g_+ = 0, \\ \sigma_- : \quad & \rho (q_t - i q_{xx} + 2i\eta |q|^2 q) + \rho_t q - \frac{1}{2} g_- = 0, \\ \sigma_3 : \quad & \eta_x |q|^2 = 0, \end{aligned} \quad (29)$$

at  $\mathcal{O}(\lambda^0)$ . From the last of the above equations, it is clear that  $\eta$  remains position-independent despite of the NH deformation. In fact it is easy to see that for the NH deformation confined only to the negative powers of the spectral parameter, this particular equation will not be effected. This system *further* satisfies the constraints of Eq.s 21 at  $\mathcal{O}(\lambda^1)$ . In addition, as we had observed in Sec. 2, the particular NH deformation in the present case induces contributions at  $\mathcal{O}(\lambda^{-1})$ , which are,

$$\begin{aligned} \sigma_3 : \quad & g_{3,x} = 2\rho^* q^* g_- - 2\rho q g_+, \\ \sigma_+ : \quad & g_{+,x} = -\rho^* q^* g_3, \\ \sigma_- : \quad & g_{-,x} = \rho q g_3. \end{aligned} \quad (30)$$

In order to obtain the deformed NLS system of Eq. 24, following first two of the Eq.s 29, the additional conditions are needed to be satisfied:

$$\begin{aligned} i\rho^* q_{xx}^* + \rho_t^* q^* + \frac{1}{2}g_+ &= i\rho^* (\eta q^*)_{xx} \equiv i\rho^* \eta q_{xx}^* \quad \text{and} \\ -i\rho q_{xx} + \rho_t q - \frac{1}{2}g_- &= -i\rho (\eta q)_{xx} \equiv -i\rho \eta q_{xx}, \end{aligned} \quad (31)$$

with the last equality coming from the last of Eq.s 29. One can always choose  $g_i$ s, subjected to the constraints 30, to satisfy the last set of equations, yielding the desired deformed NLS system. On extracting the expressions of  $g_{\pm}$  from the above equations as,

$$g_- = 2i\rho(\eta - 1)q_{xx} + 2\rho_t q \equiv -g_+^*, \quad (32)$$

and substituting back in Eq.s 30, one can re-express the NH constraints in terms of the NLS amplitude as,

$$\begin{aligned} i\rho(\eta - 1)q_{xxx} + \rho_t q_x &= 2\rho q \left[ \eta_t \int^x |q(x', t)|^2 dx' + i\eta(\eta - 1)(q^* q_x - q q_x^*) \right] \quad \text{and} \\ i\rho^*(1 - \eta)q_{xxx}^* + \rho_t^* q_x^* &= 2\rho^* q^* \left[ \eta_t \int^x |q(x', t)|^2 dx' + i\eta(\eta - 1)(q^* q_x - q q_x^*) \right], \end{aligned} \quad (33)$$

which are essentially complex-conjugates of each-other. Therefore, one can identify the temporally inhomogeneous deformation of the XXX model with a *particular* non-holonomic deformation of the NLS system (Eq.s 32) in the continuum limit, following Hasimoto's map. However, the temporal inhomogeneity deformation can persist even without this deformation, signified by the vanishing of the integrands of Eq.s 33, by virtue of Eq. 25 of the holonomic case. Further, it is easy to see that a more general non-holonomic deformation than that in Eq. 28, say of the form in Eq. 10, will yield a more extensive algebraic structure, without disturbing the foregoing conclusions. This is because the additional contributions from the extended NHD treatment will correspond to  $\mathcal{O}(\lambda^{-2})$  and lower.

### 3.2 General Parameterization: Correspondence to Inhomogeneous XXX Model

Thus far, it is clear that just by localization of the existing parameters ( $\rho$  or  $\eta$ ), even identified with an NHD, will not correspond to spatial inhomogeneity. However, it still leaves room for a more general parameterization  $|\rho(x, t)|^2 \neq \eta(x, t)$ . The different components of the zero-curvature condition get extended because of this generalization as:

$$\sigma_+ : \quad \rho^* (q_t^* + i q_{xx}^* - 2i\xi\eta|q|^2 q^*) + i\rho_x^* q_x^* + \rho_t^* q^* + \frac{1}{2}g_+ = 0,$$



$$\begin{aligned}
\sigma_- : \quad & \rho (q_t - iq_{xx} + 2i\xi\eta|q|^2q) - i\rho_x q_x + \rho_t q - \frac{1}{2}g_- = 0, \\
\sigma_3 : \quad & (\eta|q|^2)_x = |\rho|^2|q|_x^2,
\end{aligned} \tag{34}$$

at  $\mathcal{O}(\lambda^0)$ . The last of the above equations have crucially changed, as  $\rho\rho^* \neq \eta$ . This immediately leads to the result,

$$\eta|q|^2 \equiv \int_{-\infty}^x |\rho|^2 |q|_x^2 dx'. \tag{35}$$

However, this does not lead to the desired space-dependence as the *same*  $\mathcal{O}(\lambda^1)$  contribution of Eq. 21 prevails:

$$\begin{aligned}
\sigma_+ : \quad & \rho_x^* q^* = 0 \quad \text{and} \\
\sigma_- : \quad & \rho_x q = 0,
\end{aligned} \tag{36}$$

keeping  $\rho$  *space-independent*, and therefore, the modified equation,

$$q_t - i(|\rho|^2 q)_{xx} + 2iq \int_{-\infty}^x |\rho|^2 |q|_x^2 dx' = 0, \tag{37}$$

still cannot represent an inhomogeneous XXX model like that in Eq. 17 [8].

As seen in subsection 3.1, the NH deformation considered in Eq. 28 does *not* effect the set of Eq.s 36. In fact any such NH deformation with only *negative* powers of the spectral parameter  $\lambda$  cannot effect the same, as they do not contribute to the spectral domain with positive power, owing to the  $sl(2)$  loop-algebraic structure of the NLS system, implicit in the definition of the Lax pair of Eq.s 18. This is, in principle, corresponds to the spectral separation observed in case of the NH deformation of the semi-Classical limit of the HSC in subsection 2.1. Therefore, following the observation that any NH deformation to NLS of  $\mathcal{O}(\lambda^n)$  contributes at  $\mathcal{O}(\lambda^{n,n+1})$ , we further consider an  $\mathcal{O}(\lambda^0)$  deformation as,

$$\delta B = \frac{i}{2} [f_3 \sigma_3 + f_+ \sigma_+ + f_- \sigma_-]. \tag{38}$$

Such a consideration leads to the the extended set of equations from the zero-curvature condition,

$$\begin{aligned}
\sigma_+ : \quad & \rho^* (q_t^* + iq_{xx}^* - 2i\xi\eta|q|^2 q^*) + i\rho_x^* q_x^* + \rho_t^* q^* + \frac{i}{2} (f_{+,x} - \rho^* q^* f_3) = 0, \\
\sigma_- : \quad & \rho (q_t - iq_{xx} + 2i\xi\eta|q|^2 q) - i\rho_x q_x + \rho_t q + \frac{i}{2} (f_{-,x} + \rho q f_3) = 0, \\
\sigma_3 : \quad & (\eta|q|^2)_x - |\rho|^2 |q|_x^2 + \frac{1}{4} (f_{3,x} + 2\rho q f_+ - 2\rho^* q^* f_-) = 0,
\end{aligned} \tag{39}$$

at  $\mathcal{O}(\lambda^0)$ , and further,

$$\begin{aligned}
\sigma_+ : \quad & \rho_x^* q^* + \frac{1}{2} f_+ = 0 \quad \text{and} \\
\sigma_- : \quad & \rho_x q - \frac{1}{2} f_- = 0,
\end{aligned} \tag{40}$$

at  $\mathcal{O}(\lambda^1)!$ , without any contribution at  $\mathcal{O}(\lambda^{-1})$ . Therefore, we finally get rid of the explicit space-independence of  $\rho$ .

We then carry on with imposing suitable conditions on local parameters  $f_{3,\pm}$  as required.  $f_{\pm}$  have explicit expressions from Eq. 40, which was not the case in general in subsection 3.1. Then, to obtain the required integro-differential interaction term of the modified NLS system, on requiring the last bracket in the last of Eq.s 39 to vanish yields the spatial evolution of  $f_3$  as,

$$f_{3,x} = 4|\rho|_x^2 |q|^2. \tag{41}$$

Then, on finally requiring an equation of the form in Eq. 37 (or its complex-conjugate), the second (first) of Eq.s 39 leads to the condition,

$$\rho_t q - i\rho q_{xx} + i\rho_{xx} q + 2i\rho q \int_{-\infty}^x |\rho|_{x'}^2 |q|^2 dx' \equiv -i\rho (|\rho|^2 q)_{xx}. \tag{42}$$

and its complex-conjugation. The above result finally leads us to the desired integro-differential, *spatially* modified NLS system,

$$q_t - i (|\rho|^2 q)_{xx} + 2iq \int_{-\infty}^x |\rho|^2 |q|_{x'}^2 dx' = 0, \tag{43}$$

that represents the inhomogeneous XXX HSC model of Eq. 16.

In obtaining the above desired result, we exclusively required the NH deformation parameters  $f_{3,\pm}$  to be completely determined by the system parameters  $\rho$  and  $q$  (Eq.s 40 and 41), unlike the case for the parameters  $g_{3,\pm}$  in Eq. 30. This also means that the parametric constraints of Eq. 30 are replaced by Eq. 42, which imposes constraint on the original variables  $\rho$  and  $q$  of the system itself. Moreover, the self-coupling strength  $\eta$  of the original NLS system does *not* appear in the final equations, and is completely determined by the integro-differential expression of rest of the system parameters in Eq. 35, when it is spatially local also. This is another way of looking at the NH constraint, which now is translated to that of system parameters. All of this stems from the  $\mathcal{O}(\lambda^0)$  NH deformation of Eq. 38.

In principle, another NH deformation that can work is that of  $\mathcal{O}(\lambda^1)$ ,  $\propto h_i \sigma_i$ ,  $i = 3, \pm$ , since it will contribute at  $\mathcal{O}(\lambda^{1,2})$  as discussed earlier. This will contribute at the level of Eq.s 15, keeping  $\rho$  spatially non-trivial, and the corresponding constraints will come at  $\mathcal{O}(\lambda^2)$ . However, it is easy to see that those constraints precisely will be  $h_{\pm} = 0$ . Thus, only  $h_3$  will contribute in keeping  $\rho$  spatial, and will completely be determined by  $\rho$  only:

$$h_3 = 2i (\log \rho)_x. \quad (44)$$

It will yield the desired dynamics of Eq. 15, with a much simpler constraint among the system variables as,

$$\rho_t q - i\rho q_{xx} - i\rho_x q_x = -i\rho (|\rho|^2 q)_{xx}. \quad (45)$$

Therefore, any suitable combination of  $\mathcal{O}(\lambda^{0,1})$  NH deformations can work for this purpose, and it is easy to see that *no* NH deformation other orders can yield spatial inhomogeneity as it will not effect the  $\mathcal{O}(\lambda^1)$  contribution from the NLS system, making  $\rho$  purely temporal. One can trace this back to the results in subsection 2.1 of the semi-Classical HSC, specifically to Eq. 13, with the same set of conclusions, on the other side of the Hasimoto transformation. The fact that the lowest (below  $\lambda^0$ ) order deformation there was purely temporal (Eq. 15) directly corresponds to the null result of subsection 3.1 with NH deformation of Eq. 28. Thus, there is a complete partition of the spectral space in two sectors (corresponding to  $\lambda^n$  with  $n > 1$  and  $n < 0$ ) that cannot yield locality corresponding to the inhomogeneous HSC. Only the spectral patch with  $n = 0, 1$  that coincides with that of the undeformed system can yield the desired result. Further, it is easy to see that there will be a more generalized phase-amplitude entanglement in these cases, much like that in Eq. 27, reminiscent of the semi-classical origin. As the modified NLS system is integrable [9], it will be of interest to see what kind of further deformations can lead to correspondence with more complicated quantum systems, e. g., XXY and XYZ HSCs.

## 4 Discussions

A generalization of this approach can be valid in case of XXZ or XYZ HSC systems, where *anisotropy* is additionally introduced in the  $SU(2)$  subspace as,

$$\begin{aligned} H_{XXZ} &= -J \sum_{i=1, a=1}^{N, 3} [\zeta^a \mathcal{S}_i^a \cdot \mathcal{S}_{i+1}^a], \quad \zeta^1 = \zeta^2 \neq \zeta^3 \quad \text{and} \\ H_{XYZ} &= -J \sum_{i=1, a=1}^{N, 3} [\zeta^a \mathcal{S}_i^a \cdot \mathcal{S}_{i+1}^a], \quad \zeta^1 \neq \zeta^2 \neq \zeta^3. \end{aligned} \quad (46)$$

Then one can define, in the semi-classical limit,

$$S = \begin{pmatrix} \zeta^3 t_3 & \zeta^1 t_1 - i\zeta^2 t_2 \\ \zeta^1 t_1 + i\zeta^2 t_2 & -\zeta^3 t_3 \end{pmatrix} \equiv \sum_{i=1}^3 T_i \sigma_i, \quad T_i = \zeta^i t_i. \quad (47)$$

Instead of  $\mathbf{t}$ , if we identify  $\mathbf{T}$  as the Frenet-Serret tangent vector of unit magnitude, apart from an additional, but internal, constraint of parameters  $\zeta^i$ 's, the usual procedure considered in this manuscript will go through, *without* any distinction regarding the presence of the  $S(2)$  anisotropy. This is expected following decoupled nature of the

group space ( $\mathcal{M} = \mathbb{R} \otimes SU(2)$ ) in the complete manifold  $\mathcal{M}$ . Indeed, this anisotropy can be tapped by considering  $\mathbf{t}$  as the Frenet-Serret tangent. However, as a down-side, the corresponding geometry harbors that anisotropy, prohibiting the construction of the Hasimoto transformation of Eq. 4. We leave this possibility for the future.

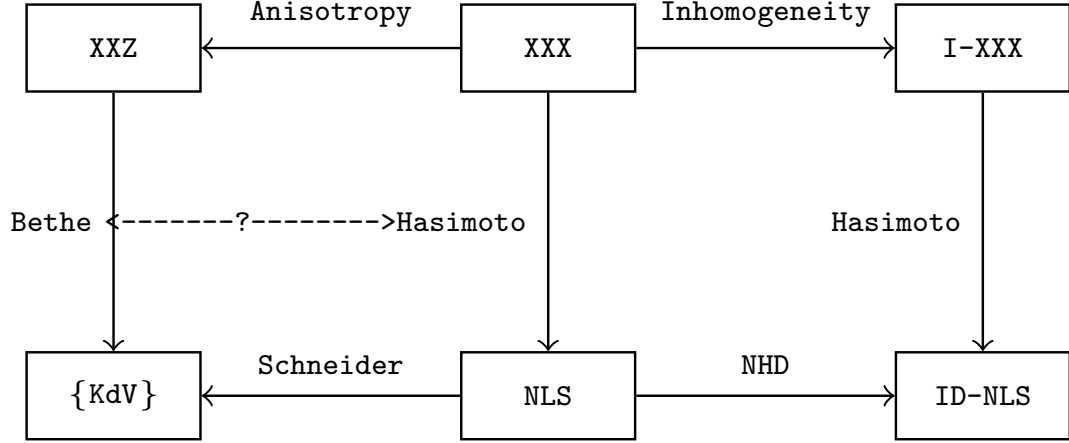


Figure 1: Schematics of HSC correspondence with continuum integrable systems. Here I-XXX stands for *inhomogeneous XXX* model, and ID-NLS for *Integro-differential NLS* system. The right half of the diagram represents the present results, connecting inhomogeneity of the discrete case with the integro-differential generalization of the continuum (NLS) sector, whereas the left half summarizes the possibilities regarding *anisotropy* in the  $SU(2)$  subspace. The braces about KdV marks the extended KdV family, including quantum (modified) KdV systems. The generalization of the Hasimoto map, mentioned in Section 4 for the anisotropic XXX model, could be connected to the Bethe ansatz approach linking quantum m-KdV and continuum XXZ model.

Another interesting aspect is to exploit the weak correspondence that exists through the Schneider map [15, 16] between NLS and KdV amplitudes, when the coupling-strength for the non-linearity is small. It will be of interest to see what form of generalization takes place in the KdV side, following Eq. 43, and how it is related to the usual KdV systems. The recent observation of complementing quasi-integrability in both sides of the Schneider map [17] intuitively supports the expectation that the NH deformation representing the inhomogeneous system will correspond to a particular NH deformation of the KdV system. Moreover, as the Bethe Ansatz equation for the quantum (modified) KdV equation [18, 19, 20] can be viewed as the continuous limit of the XXZ model [16], the above-mentioned generalized Hasimoto transformation for the anisotropic case may lead to KdV for the XXZ case, instead of NLS system. In that case, the Bethe Ansatz approach could be related to the Hasimoto map by considering the Schneider map. A schematic representation of these interconnected structure is given in Fig. 1.

## 5 Conclusion

We have shown, through explicit analysis, that the Hasimoto analogue of continuum inhomogeneous XXX HSC is a particular NH deformation of the NLS system, the latter itself being the Hasimoto analogue of continuum homogeneous XXX HSC model. This identification crucially depends on the spectral range ( $\lambda^0$  and  $\lambda^1$ ) of the NLS dynamics, which doubles as the same for continuum HSC, thereby splitting the spectrum in two parts, that can correspond to temporal inhomogeneity at best. These deformed NLS systems can be interpreted in terms of the deformations in the corresponding Frenet-Serret manifold. They further display amplitude-phase dependence, absent for the usual NLS system, bearing semi-classical signature. This is expected as the usual XXX HSC has a classical mean-field description, not possible in presence of local inhomogeneity at lattice sites. The precise NH mapping from NLS to the inhomogeneous NLS is supported further by integrability of the latter system [9].

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